MATH 7350 HW 1 - Geometry of Manifolds

Hossien Sahebjame; ID # 2188945

September, 2023

1.

Give an example of a manifold that has all properties except for Hausdorffness.

Proof. Let
$$X = \mathbb{R} \times \{-1\} \cup \mathbb{R} \times \{1\} \subset \mathbb{R}^2$$
be a subset of the plane. Define $\sim$ on X where
$$(x, -1) \sim (x, 1)$$
for all $x \neq 0$. I claim $M = X/\sim$ is a smooth 2nd countable manifold which is non-Hausdorff. As M is a subspace of a second countable space, it is itself second countable. For non-Hausdorff property, first let
$$\pi : X \to M$$
be the natural projection map defined via
$$x \mapsto [x].$$
Then, note if $x \neq 0$ then $(x, -1) \sim (x, 1)$ so suppose $x \neq 0$ to obtain 2 distinct points of M. So let $[(0, -1)], [(0, 1)], \in U, V$ respectively, with $U, V \subset M$ open. By continuity of π, we have that
$$\pi^{-1}(U), \pi^{-1}(V) \subset X$$
are open and contain $(0, -1)$ and $(0, 1)$ respectively. So there exists an $\epsilon > 0$ such that
$$(-\epsilon, \epsilon) \times \{-1\} \subset \pi^{-1}(U), (-\epsilon, \epsilon) \times \{1\} \subset \pi^{-1}(V).$$
Thus
$$\pi((-\epsilon, \epsilon) \times \{-1\}) \subset U, \pi((-\epsilon, \epsilon) \times \{1\}) \subset V$$
and by definition of π, $U \cap V \neq \emptyset$. To show the locally Euclidean property, let π be as above and let $[(x, 1)] \in M$. By continuity of π, there exists some open set
$$(a, b) \times \{1\} \subset X$$
such that $x \in (a, b)$. I claim the map
$$\phi : \pi((a, b) \times \{1\}) \to (a, b)$$

defined via

$$[(y, \pm 1)] \mapsto y$$

is the desired homeomorphism. It is clearly onto by contrsuction. And if $[(y, \pm 1)] = [(y', \pm 1)]$ then $y = y'$ thus the mapping it well-defined. For 1-1, suppose

$$\phi([y, \pm 1]) = \phi([y', \pm 1]).$$

If $y = y' = 0$, then

$$[y, \pm 1] = [y', \pm 1] = [0, 1]$$

as $[y, \pm 1], [y', \pm 1] \in \pi((a, b) \times \{1\})$. Otherwise, we can ignore the ± 1 since it does not affect the equivalence class thus

$$[y, \pm 1] = [y', \pm 1].$$

Therefore ϕ is 1-1. For locally Euclidean, let $U \subset (a, b)$ be open. Then

$$\phi^{-1}(U) = \phi(U \times \{1\}).$$

And since

$$\pi^{-1}(\pi(U \times \{1\})) = U \times \{\pm 1\} \subset X$$

is open, $\phi^{-1}(U)$ is open. $\square$

2.

Give an example of a manifold that has all properties except for 2nd countability.

Proof. Consider the manifold

$$M = \mathbb{R}^{[0,1]}.$$

That is, M is the uncountable product of the reals. This space is Euclidean and thus has smooth structure. Furthermore as $\mathbb{R}$ is Hausdorff, an arbitrary product is still Hausdorff, however as we have an uncountable product there cannot exists a countable basis thus M is not second countable. $\square$

3.

Prove $\mathbb{P}^n$ is Hausdorff.

Proof. There is a lemma from point set topology which I took and passed the prelim in, that states the graph of a space is closed if and only if the space is Hausdorff. So I claim

$$\Gamma := \{(x, y) : x, y \in \mathbb{R}^{n+1} \setminus \{0\}, : y = kx \text{ for some } k \in \mathbb{R}^{\times}\}$$

is closed. Let $(x, y) \in \Gamma$. Then we can write $y = kx$, for some $k \in \mathbb{R}^{\times}$. I.e., we have (x, kx). In other words the matrix whose columns are x and y have rank less than or equal to 1. This is equivalent to vanishing space of this matrix which is a closed set, thus Γ is closed forcing $\mathbb{P}^n$ to be Hausdorff.

□

4.

Consider $\mathbb{R}$ as a topological manifold in the usual sense. Prove there exists uncountably many smooth structures on $\mathbb{R}$.

Proof. By the hint given in Lee, for $s > 1$ consider

$$F_s(x) := |x|^{s-1}x.$$

And I claim F_s is a homemorphism from the n ball to itself and a diffeomorphism if and only if $s = 1$. And as the 1 ball is homeomorphic to $\mathbb{R}$ this will suffices. Note F_s is not defined at $x = 0$ for $s \leq 1$. However, $|F_s| = |x|^s$ thus defining $F_s(0) = 0$ makes F_s continuous. More over, the norm of F_s is less than or equal to 1 thus the image of F_s lies in the 1 ball. Furthermore, its inverse is given as

$$F_s(x)^{-1} := |x|^{\frac{1}{s}-1}x$$

where $F_s^{-1}(0) = 0$ thus F_s^{-1} is continuous thus F_s is a homeomorphism for each $s > 0$. Also, away from $x = 0$, F_s is a diffeomorphism as F_s, F_s^{-1} are products and quotients of smooth nonvanishing functions. Moreover, F_1 is the identity and thus a diffeomorphism. Thus as $s > 0$ ranges over the positive reals, $\mathbb{R}$ has uncountably many smooth structures. $\square$

5.

Let M be a smooth manifold, let $f : M \to \mathbb{R}^k$ be a smooth function, prove $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$ is smooth for every chart (U, ϕ) contained in the maximal atlas defining M.

Proof. Let M be a smooth manifold with $f : M \to \mathbb{R}^k$ smooth. Let (U, ϕ) be arbitrary with $p \in U$. As f is smooth, there exists a chart (V, ψ) with $p \in V$ such that

$$f \circ \psi^{-1} : \psi(V) \to \mathbb{R}^k$$

is smooth. However we know transition function

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is smooth. Thus the composition

$$(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1}) = f \circ \phi^{-1} : \phi(U \cap V) \to \mathbb{R}^k$$

is smooth and $p \in U$ was arbitrary thus $f \circ \phi^{-1}$ is smooth on the whole of $\phi(U)$. □

6.

Let $F : M \to N, G : N \to P$ be smooth maps between manifolds, show $G \circ F : M \to P$ is a smooth map between manifolds.

Proof. As F, G are smooth, for every $p \in M$, with $F(p) \in N, G(F(p)) \in P$ there exists charts on them say $(U, \phi), (V, \psi), (W, \theta)$ such that

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

and

$$\theta \circ G \circ \psi^{-1} : \psi(V) \to \theta(W).$$

are both smooth. Thus

$$\begin{aligned} \theta \circ G \circ \psi^{-1} \circ \psi \circ F \circ \phi^{-1} &: \phi(U) \to \psi(V) \\ = \theta \circ G \circ F \circ \phi^{-1} &: \phi(U) \to \theta(W) \end{aligned}$$

is smooth thus $G \circ F$ is smooth as there exists charts $(U, \phi), (W, \theta)$ such that

$$\theta \circ G \circ F \circ \phi^{-1}$$

is smooth. □

7.

Let M be a smooth manifold of degree at least 1. Prove $C^\infty(M)$ is an infinite dimensional vector space.

Proof. We must show for every $f, g, h \in C^\infty(M)$ and all $a, b \in \mathbb{R}$ that

1. $f + g \in C^\infty(M)$ (closuse with respect to addition)
2. $af \in C^\infty(M)$ (closure with respect to scalar multiplication)
3. $(f + g) + h = f + (g + h)$ (associativity of addition)
4. $f + g = g + f$ (commutativity of addition)
5. There exists an element $0 \in C^\infty(M)$ such that $f + 0 = f$ (existence of additive identity)
6. There exists an inverse $f^{-1} \in C^\infty(M)$ such that $f + f^{-1} = 0$. (existence of inverse elements)
7. $a(bf) = (ab)f$ (scalar multiplication)
8. $1f = f$ (multiplicative identity)
9. $a(f + g) = af + ag$ (distribution of scalars)
10. $(a + b)f = af + bf$ (scalar multiplication with respect to field addition)

And that the dimension of $\mathbb{C}^\infty(M)$ is ∞. For (1), let $f, g \in C^\infty(M)$. Then there exists a chart (U, ϕ) such that

$$f \circ \phi^{-1}, g \circ \phi^{-1}$$

are both smooth. Then

$$(f \circ \phi^{-1}) + (g \circ \phi^{-1}) = (f + g) \circ \phi^{-1}$$

is smooth thus $f + g \in C^\infty(M)$. For (2), let $a \in \mathbb{R}, f \in C^\infty(M)$. We seek to show $af \in C^\infty(M)$. Define

$$g(x) := af(x).$$

Its enough to show

$$g \circ \phi^{-1}$$

is smooth on $\phi(U)$. And as smoothness is translation invariant the above map is smooth thus $af \in C^\infty(M)$. Then (3) and (4) follow as f, g, h are real valued and $\mathbb{R}^n$ is an abelian group under $+$. For (5) define thee zero function as

$$\zeta(p) = 0; \qquad \text{for all } p \in M.$$

Then it is clear that for any $f \in C^\infty(M)$,

$$f + \zeta = f + 0 = f.$$

and we have (5). For (6), if $f \in C^\infty(M)$, then the additive inverse is $-f$ which is in $C^\infty(M)$ by (2). And it is clear that

$$f(x) + -f(x) = 0.$$

and we have (6). Then (7) follows from (2). Item (8) follows from the fact that 1 is the multiplicative identity in $\mathbb{R}$ and $f(x) \in \mathbb{R}^n$. Thus $1f(x) = f(x)$. Lastly, (9) and (10) follow as $f, g \in \mathbb{R}^n$ are real-valued and the reals satisfy distribution rules. Thus $C^\infty(M)$ is a vector space. I now claim its dimension is ∞. Let (U, ϕ) be an arbitrary chart of M. Then

$$\phi : U \to \mathbb{R}^n.$$

Let $k \in \mathbb{Z}^+$ and let $x_1, ..., x_k \in \phi(U)$ be distinct points. By Hausdorff property of $\mathbb{R}^n$, for each x_j there exists disjoint open set V_j and ϵ_j such that

$$B_j := \overline{B_{\epsilon_j}(x_j)} \subset V_j.$$

If

$$f_j : \mathbb{R}^n \to \mathbb{R}$$

are bump functions, which exists by Theorem 2.25 of Lee, defined on each B_j and $W_j := \phi^{-1}(V_j)$, then define the continuous functions

$$g_j : M \to \mathbb{R}$$

via

$$g_j(x) = \begin{cases} f_j(\phi(x)) & \text{if } x \in W_j \\ 0 & \text{otherwise} \end{cases}$$

which may be extended to smooth functions $\tilde{g}_j$ on M such that $g_j|_{W_j} = \tilde{g}_j|_{W_j}$. And these $\tilde{g}_j$ are linearly independant. □

8.

Prove that if G is a smooth manifold with group structure such that the map $f : (g, h) \mapsto gh^{-1}$ is smooth, then G is a Lie groups.

Proof. We already have that G is smooth manifold with group structure. Let $g, h \in G$. It suffices to show the maps

$$m(g, h) = gh$$

and

$$\iota(g) = g^{-1}$$

are smooth. First off observe that $f(g, h) = m(g, \iota(h))$. Well note

$$\begin{aligned} f(e, h) &= eh^{-1} && \text{definition of } f \\ &= \iota(h) && \text{definition of } \iota \end{aligned}$$

Thus ι is smooth as f is smooth. Then note

$$\begin{aligned} f(g, h) &= f(g, \iota(\iota(h)) && \iota(\iota)\text{=id}_G \\ &= gh && \text{definition of } f \\ &= m(g, h) && \text{definition of } m \end{aligned}$$

then m is smooth as f is, thus G is a lie group. □

9.

Let M be a topological space such that for any open cover $\mathcal{X}$ of M there exists a partition of unity subordinate to $\mathcal{X}$. Prove M is paracompact.

Proof. Let $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ be an arbitrary open covering for M. That is,

$$M = \bigcup_{\alpha \in A} X_\alpha.$$

Here A is some arbitrary indexing set. I claim $\mathcal{X}$ admits a locally finite refinement making M paracompact. By assumption, there exists a partition of unity subordinate to $\mathcal{X}$. This means there exists a family $\{\psi_\alpha\}_{\alpha \in A}$ of continuous functions

$$\psi_\alpha : M \to \mathbb{R}$$

such that $\psi_\alpha(x) \in [0,1]$ for every $\alpha \in A$ and all $x \in M$, supp $\psi_\alpha \subset X_\alpha$ for all $\alpha \in A$, and $\mathcal{S} := \{\text{supp } \psi_\alpha\}_{\alpha \in A}$ is locally finite, and

$$\sum_{\alpha \in A} \psi_\alpha(x) = 1$$

for every $x \in M$. Now define

$$A' := \{\alpha : \alpha \in A, \psi_\alpha \neq 0\}.$$

Then take $Y_\alpha := \psi_\alpha^{-1}((0,1])$, then $Y := \{Y_\alpha : \alpha \in A'\}$ is the desired locally finite refinement of $\mathcal{X}$, since $\mathcal{S}$ is locally finite, Y is locally finite. And as supp $\psi_\alpha \subset X_\alpha$, Y is a refinement. Thus M is locally compact. ☐

10.

Let M be a smooth manifold. Let $B \subset M$ be closed and suppose $\delta : M \to \mathbb{R}^+$ be continuous.

(a) Using partitions of unity, prove there exists a smooth function $\tilde{\delta} : M \to \mathbb{R}^+$ such that $\tilde{\delta}(x) \in (0, \delta(x))$ for all $x \in M$.

(b) Prove there exists a continuous function $\psi : M \to \mathbb{R}$ that is smooth and positive on $M \setminus B$, identically equal to zero on B, and satisfies $\psi(x) < \delta(x)$ everywhere on M.

Proof. (a) As M is a smooth manifold, if $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ is any arbitrary open covering of M, then by Theorem 2.23 of Lee, there exists a smooth partition of unity subordinate to $\mathcal{X}$. I.e., there exists a family of smooth functions $\{\psi_\alpha\}_{\alpha \in A}$ such that

1. $\psi_\alpha(x) \in [0, 1]$ for every $x \in M$, and every $\alpha \in A$.
2. supp $\psi_\alpha \subset X_\alpha$ for each $\alpha \in A$.
3. $\{\text{supp } \psi_\alpha\}_{\alpha \in A}$ is locally finite.
4. $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for every $x \in M$.

Furthermore, by our assumption suppose $\delta : M \to \mathbb{R}^+$ is a continuous mapping. For each $p \in M$, let $U_p \subset M$ be open and choose $\epsilon_p > 0$ such that

$$\epsilon_p < \inf \delta(U_p).$$

Since M is paracompact, there exists an open covering $\{W_i\}_{i \in I}$ such that for each $p \in M$ we can choose $i \in I$ such that $W_i \subset U_p$. then put $\delta_i := \epsilon_p$, then on W_i we have $\delta_i < \inf \delta(W_i)$. Thus putting

$$\tilde{\delta}(x) := \sum \delta_i \psi_i(x)$$

works and we have $\tilde{\delta}(x) \in (0, \delta(x))$ as needed. Furthermore, $\tilde{\delta}$ is continuous as it is a sum of continuous functions with scalars.

(b) By the hint let

$$f : M \setminus B \to \mathbb{R}$$

be a positive exhaustion function define via $f(x) := \delta(x)$ if $\delta(x) > 1$ and $\frac{1}{\delta(x)}$ if $\delta(x) < 1$. By Theorem 2.29 of Lee, since $B \subset M$ is closed, there exists a smooth function

$$g : M \to \mathbb{R}$$

such that $g(B) = \{0\}$. Then define

$$\psi : M \to \mathbb{R}$$

via

$$\psi(x) := \begin{cases} \frac{1}{1+f(x)} & x \in M \\ g(x) & x \in B \end{cases}$$

which is smooth at both f, g are smooth and it satisfies

$$\psi(x) < \delta(x)$$

for every $x \in M$. Also ψ is positive everywhere on $M \setminus B$ and zero on B. ☐