

# MATH 7350 HW 3 - Geometry of Manifolds

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## 1.

Let  $\mathfrak{g}$  be a Lie algebra. A linear subspace  $\mathfrak{h} \subseteq \mathfrak{g}$  is said to be an *ideal* if for every  $X \in \mathfrak{g}, Y \in \mathfrak{h}$ , one has  $[X, Y] \in \mathfrak{h}$ .

(i) Explain why every ideal in  $\mathfrak{g}$  is a Lie subalgebra in  $\mathfrak{g}$ .

(ii) If  $\mathfrak{h}$  is an ideal in  $\mathfrak{g}$ , prove the quotient vector space  $\mathfrak{g}/\mathfrak{h}$  has a unique Lie algebra structure such that the canonical projection

$$\pi : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$$

is a Lie algebra homomorphism.

*Proof.* (i) To show  $\mathfrak{h} \subset \mathfrak{g}$  is a Lie subalgebra, we must verify  $\mathfrak{h}$  is closed under the bracket operation, as it is already a linear subspace. Let  $X, Y \in \mathfrak{h}$ , then  $Y \in \mathfrak{g}$  and since  $\mathfrak{h}$  is an ideal, we have that  $[X, Y] \in \mathfrak{h}$  thus  $\mathfrak{h} \subset \mathfrak{g}$  is a Lie subalgebra.

(ii) We must show that the bracket operation under this mapping is unique, for  $\pi : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$  to be a homomorphism. As  $\pi$  is the canonical projection, we have that for every  $g \in \mathfrak{g}$ , the map sends

$$g \mapsto g + \mathfrak{h}.$$

Here,  $g + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$  is an equivalence class. If  $[\cdot]_{\mathfrak{g}}$  denotes the Lie bracket taken in  $\mathfrak{g}$  and  $[\cdot]_{\mathfrak{g}/\mathfrak{h}}$  denotes the Lie bracket taken in the quotient space, then we have that  $\pi$  is a homomorphism if and only if for every  $X, Y \in \mathfrak{g}$ ,

$$\pi[X, Y]_{\mathfrak{g}} = [\pi(X), \pi(Y)]_{\mathfrak{g}/\mathfrak{h}}.$$

And this holds if and only if

$$[X + \mathfrak{h}, Y + \mathfrak{h}]_{\mathfrak{g}/\mathfrak{h}} = [X, Y]_{\mathfrak{g}} + \mathfrak{h}.$$

All that is left to show is for all  $X_1, X_2, Y_1, Y_2 \in \mathfrak{g}$  that if

$$X_1 + \mathfrak{h} = X_2 + \mathfrak{h}$$

and

$$Y_1 + \mathfrak{h} = Y_2 + \mathfrak{h},$$

then

$$[X_1, Y_1] + \mathfrak{h} = [X_2, Y_2] + \mathfrak{h} \quad (*).$$

By arithmetic in cosets, we have that if  $X_1 + \mathfrak{h} = X_2 + \mathfrak{h}$  and  $Y_1 + \mathfrak{h} = Y_2 + \mathfrak{h}$ , then

$$X_1 - X_2, Y_1 - Y_2 \in \mathfrak{h}.$$

And in order for (\*) to hold, we must have

$$[X_1, Y_1] - [X_2, Y_2] \in \mathfrak{h}.$$

But this holds as  $X_1 - X_2, Y_1 - Y_2 \in \mathfrak{h}$ .

□

## 2.

Let  $E$  be a vector bundle over a topological space  $M$ . Prove the projection

$$\pi : E \rightarrow M$$

is a homotopy equivalence.

*Proof.* Let  $z$  be the zero section, I claim  $\pi \circ z = \text{Id}_M$ . Let  $p \in M$  and  $v \in E_p$ . Then the mapping  $\pi_t(v) = tv$  is well-defined. Furthermore,  $\pi_1 = \text{Id}_M$  and

$$z \circ \pi = \pi_0.$$

I.e.,  $z \circ \pi = \text{Id}_M$ . Therefore  $\pi$  is a homotopy equivalence. □

### 3.

Let  $M$  be a smooth connected manifold. Show any 2 points in  $M$  can be joined by a piecewise smooth curve segment.

*Proof.* Let  $p \in M$  be arbitrary. And define

$$\mathcal{C} := \{q \in M : \text{there is a piecewise smooth curve segment in } M \text{ from } p \text{ to } q\}.$$

Clearly  $\mathcal{C} \neq \emptyset$  as  $p \in \mathcal{C}$ . I claim  $\mathcal{C} = M$  by showing it is both open and closed in  $M$ .

For open, let  $q \in \mathcal{C}$  be arbitrary. So there exists some  $\gamma$  from  $p$  to  $q$ . Let  $U$  be a smooth coordinate ball centered at  $q$ , if  $q' \in U$ , then one can go from  $p$  to  $q'$  by first following along  $\gamma$  from  $p$  to  $q$  and using the straight line homotopy from  $q$  to  $q'$  forcing  $U \subset \mathcal{C}$  thus  $\mathcal{C}$  is open.

For closed, let  $q \in \partial\mathcal{C}$  and let  $U$  be as from above. The fact that  $q$  is a boundary point means there exists some  $q' \in U \cap \mathcal{C}$ , in this case we go from  $p$  to  $q'$  then a straight line in coordinates from  $q'$  to  $q$  forcing  $q \in \mathcal{C}$  thus  $\mathcal{C} \subset M$  is closed therefore is the whole of  $M$  and we are done. □

#### 4.

Let  $F : M \rightarrow N$  be smooth and  $\gamma : [a, b] \rightarrow M$  a smooth curve segment. If  $\eta$  is a smooth cotangent vector field on  $N$ , prove

$$\int_{\gamma} F^* \eta = \int_{F \circ \gamma} \eta.$$

*Proof.* Note we have,

$$\begin{aligned} \int_{F \circ \gamma} \eta &= \int_{[a, b]} (F \circ \gamma)^* \eta && \text{definition} \\ &= \int_{[a, b]} \gamma^* F^* \eta && \text{property of } * \\ &= \int_{\gamma} F^* \eta && \text{definition} \end{aligned}$$

as needed. □

## 5.

Consider the following cotangent vector fields on  $\mathbb{R}^3$ :

$$\omega = -\frac{4z}{(x^2 + 1)^2} dx + \frac{2y}{y^2 + 1} dy + \frac{2x}{x^2 + 1} dz,$$

$$\eta = -\frac{4xz}{(x^2 + 1)^2} dx + \frac{2y}{y^2 + 1} dy + \frac{2}{x^2 + 1} dz.$$

(i) Evaluate each integral on the straight line from  $(0, 0, 0)$  to  $(1, 1, 1)$ .

(ii) For each, determine if exact.

(iii) Find the potential function if exact, and use to recompute (i).

*Proof.* Let our curve be

$$\gamma : [0, 1] \rightarrow \mathbb{R}^3$$

via

$$t \mapsto (t, t, t).$$

Then we have

$$\int_{\gamma} \omega = \int_{[0,1]} \gamma^* \omega.$$

Thus

$$\begin{aligned} \int_{\gamma} \omega &= \int_{[0,1]} \gamma^* \omega \\ &= \int_0^1 -\frac{4t}{(t^2 + 1)^2} dt + \int_0^1 \frac{2t}{t^2 + 1} dt + \int_0^1 \frac{2t}{t^2 + 1} dt \\ &= \int_0^1 -\frac{4t}{(t^2 + 1)^2} dt + 2 \int_0^1 \frac{2t}{t^2 + 1} dt \\ &= -2 \int_0^1 \frac{1}{u^2} du + 2 \int_0^1 \frac{1}{u} du \\ &= -2 \left( -\frac{1}{t^2 + 1} \right) \Big|_0^1 + 2 \ln(t^2 + 1) \Big|_0^1 \\ &= -1 + 2 \ln(2). \end{aligned}$$

Now for  $\eta$ ,

$$\begin{aligned} \int_{\gamma} \eta &= \int_{[0,1]} \gamma^* \eta \\ &= - \int_0^1 \frac{4t^2}{(t^2 + 1)^2} dt + \int_0^1 \frac{2t}{t^2 + 1} dt + 2 \int_0^1 \frac{1}{t^2 + 1} dt \\ &= -\frac{\pi - 2}{2} + \ln(2) + \frac{\pi}{2} \\ &= 1 + \ln(2). \end{aligned}$$

(ii) I claim  $\omega$  is not exact, but  $\eta$  is exact, to see the potential function, see part (iii). To show  $\omega$  is not exact, it suffices to show it is not closed. To see this, note

$$\begin{aligned}\frac{\partial}{\partial z} \left( -\frac{4z}{(x^2+1)^2} \right) &= -\frac{4}{(x^2+1)^2} \\ &\neq \frac{\partial}{\partial x} \left( \frac{2x}{x^2+1} \right) \\ &= \frac{2-2x^2}{(x^2+1)^2}.\end{aligned}$$

I.e.,

$$\frac{\partial \omega_x}{\partial z} \neq \frac{\partial \omega_z}{\partial x}.$$

Thus  $\omega$  is not closed and therefore it cannot be exact because if exact, then its closed.

(iii) To see why  $\eta$  is exact, the potential is

$$f(x, y, z) = \ln(y^2 + 1) + \frac{2z}{x^2 + 1}.$$

Therefore  $\eta$  is exact as we found a  $C^\infty$  function such that its derivative is  $\omega$ . It is clear to see that

$$\eta = df.$$

□

**6.**