My (2) Cents on Differential Geometry: From Charts ⇒ Tangent Bundles ⇒ Differential Forms & Stokes' Theorem

Professor Hossien Sahebjame, M.S.

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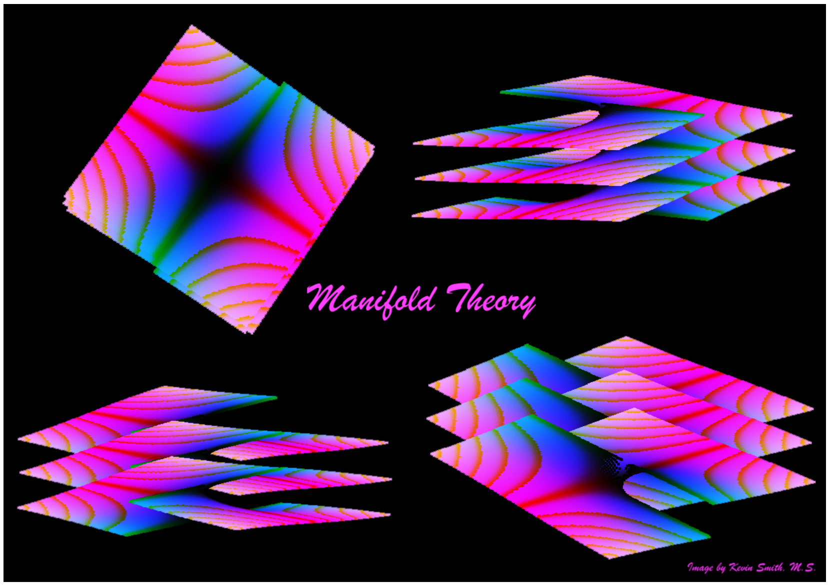
Preface:

In this introductory set of notes on differential geometry we assume the reader is familiar with multi-variable calculus, undergraduate real analysis, point set topology and basics of group theory & set theory. We aim to introduce topics such as smooth manifolds, smooth maps between them, tangent spaces, differential forms, and end with Stokes Theorem and a brief intro to the de Rham cohomology. Continuity is understood to be either epsilon delta or open set definition as all smooth manifolds are metric spaces thus these definitions coincide.



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§ 0. Frequently Used Spaces/Symbols; Notation

1. $\mathbb{R}^n, \mathbb{C}^n$ denote the n dimensional reals and complex numbers.
2. $\mathbb{Z}^{\geq 0} = \{0, 1, 1, ...\}$.
3. $\mathbb{Z}^{+}\{1, 2, 3, ...\}$.
4. Similarly, $\mathbb{R}^{+}, \mathbb{R}^{\geq 0}$ will denote the positive and non-negative reals, respectively.
5. $\mathbb{H}^n$ the upper half plane is $\mathbb{H}^n := \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_j > 0\}$, and its closure $\overline{\mathbb{H}^n} = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_j \geq 0\}$, its boundary is the $x-$axis. The closure is an example of a smooth manifold with boundary.
6. $\mathbb{RP}^n \subset \mathbb{R}^{n+1} \setminus \{\bar{0}\}$ is the real projective space defined as

   $$\mathbb{RP}^n := \{[\bar{x}] \in \mathbb{R}^{n+1} \setminus \{\bar{0}\} : [\bar{x}] \sim [\bar{y}] \text{ if and only if } \bar{x} = \lambda \bar{y}, \text{ for some } \lambda\}.$$

   that is, the equivalence class of lines through the origin or the 1 dimensional vector sub-space of $\mathbb{R}^{n+1}$.
7. $\mathbb{S}^n = \{(x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1} : \sum_{j=0}^{n} x_j^2 = 1\}$.
8. The notation $\sum_j$ is shorthand for $\sum_{j \in \mathbb{N}}$.
9. When we write $\tilde{U}$, we typically are referring to an element of the topology on $\mathbb{R}^n$. Whereas U is an element of the topology on the given manifold, M. I.e., $U \in \tau_M, \tilde{U} \in \tau_{\mathbb{R}^n}$.

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§ 1. What are manifolds?

Side-Note: We assume the reader is familiar with point set topology and if we write τ_X we refer to the topology on the set X. All manifolds are assumed to have an arbitrary (well countable) open coverings by elements of the topology. Furthermore, all manifold structures shown in these notes are assumed to have real structure as apposed to complex which in the 2 real or 1 complex dimension case gives rise to what is known as a Riemann surface or an algebraic curve.

First some definitions.

Definition 1.1 A topological space (X, τ) is said to be T_1 or *Hausdorff*, if for any $x \neq y \in X$, there exists $U, V \in \tau$ with $x \in U, y \in V$ such that $U \cap V = \emptyset$.

Examples:

1. Any metric space is Hausdorff by taking the balls around distinct points to be half the distance between them and use triangle inequality.
2. The line with two origins is not Hausdorff and neither is the Cofinite topology. I.e., if(X, τ_{cf}) is the cofinite (in algebraic geometry referred to as the Zariski Topology) this means $U \in \tau_{\text{cf}}$ means that $X \setminus U = \{x_1, x_2, ..., x_n\}$, i.e., its complement in the ambient space is a finite set, this space is also not Hausdorff if X has infinite cardinality (not measure).

Definition 1.2 A topological space (X, τ) is said to be *second-countable* if there exists a countable basis for the topology.

Examples:

1. The reals are second countable as you can cover them with a union of intervals of the form (a_j, b_j), $a_j, b_j \in \mathbb{Q}$ for all j.
2. If you take even unit interval $I = [0, 1]$, and raise this to the power of the cardinality of $\mathbb{R}$, it wont be second countable, i.e., $[0, 1]^{\aleph_1}$ i.e., $[0, 1]^{|\mathbb{R}|}$ is not second countable.

§ 1.1: Smooth Manifolds, Atlases & Charts

We skip right to the definition of a smooth (as apposed to topological) manifold. Loosely speaking an $n-$dimensional smooth manifold locally (if you zoom in to a point) acts like $\mathbb{R}^n$ but globally is not the whole of $\mathbb{R}^n$.

Definition 1.3 (manifold, charts, atlases) In other words, a *smooth $n-$manifold* is a $2-$nd countable Hausdorff topological space M (though some authors, very few omit the 2nd countability and/or Hausdorff property) with an *atlas*, denoted $\mathcal{A}$ which is the collection $\{\phi_a, U_a\}_{a \in A}$ of *charts* with $\{U_a\}_{a \in A}$ an open cover for M. For each $a \in A$, one has

$$\phi_a : U_a \to \tilde{U},$$

where $\tilde{U} \subset \mathbb{R}^n$ is open is a homeomorphism. With the additional property that if $(\phi_a, U_a), (\phi_b, U_b)$ are two charts whose domains overlap, then the *transition map*,

$$\phi_b \circ \phi_a^{-1} : \phi_a(U_a \cap U_b) \to \phi_b(U_a \cap U_b),$$

is a homeomorphism. Using *Zorn's Lemma* we can show every manifold has a maximal atlas.

We omit details such as manifold with or without boundary or orientable vs non-orientable as most key results hold for any manifold regardless of boundary or orientability. It is a well known fact that the boundary of an $n-$manifold is an $(n-1)-$manifold.

Example 1.1 Consider the unit circle, $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, it is a smooth $1-$manifold.

Proof. I claim that two charts suffice to prove S^1 is a $1-$Manifold. Let $U_1 := S^1 \setminus \{(0, 1)\}$ and $U_2 := S^1 \setminus \{(0, -1)\}$. I.e., we removed the north and south poles respectively. So I will let you figure out what ϕ_1, ϕ_2 are if

$$\phi_1^{-1}(\theta) = (\cos\theta, \sin\theta); \qquad \theta \in (-\pi, \pi).$$

and

$$\pi_1(U_1) = (-\pi, \pi).$$

Similarly,

$$\phi_2^{-1}(\theta) = (\cos\theta, \sin\theta); \qquad \theta \in (0, 2\pi).$$

and

$$\phi_2(U_2) = (0, 2\pi).$$

And $U_1 \cup U_2 = S^1$. Now on the overlaps, $U_1 \cap U_2$, define the transition maps via

$$\phi_2 \circ \phi_1^{-1}(\theta) := \begin{cases} \theta + 2\pi & \theta < 0 \\ \theta & \theta > 0. \end{cases}$$

where

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2).$$

So transition maps are smooth as well. □

Example 1.2 The classical example, $\mathbb{R}^2$ itself is a $2-$ manifold.

Proof. To see this, A single chart will suffice, namely $(\phi, \mathbb{R}^2)$ so the $U := \mathbb{R}^2$ itself and

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2$$

is defined via

$$\phi((x, y)) := (x, y).$$

□

Other examples include but are not limited to the general linear group, higher dimensional spheres, upper half plane, level set curves, etc.

Theorem 1.2: Topological invariance of dimension $\mathbb{R}^m \cong \mathbb{R}^n$ if and only if $m = n$.

We will give a proof of this after we have developed a bit more machinery (the de Rham cohomology will be the tool which is used). It is a consequence of the *Invariance of Domain*.

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§ 2. Smooth Maps

§ 2.1: Smooth Maps between manifolds

Before we introduce the notion of morphisms (category theory language) we will introduce the notion of a smooth mapping or function on smooth manifolds.

Definition 2.1 Let M be a smooth $n-$manifold ans let $k \in \mathbb{Z}^{\geq 0}$, and

$$f : M \to \mathbb{R}^k$$

is any function. We say f is a *smooth function* if for all $p \in M$, there exists a smooth chart (ϕ, U) for M with $p \in U$ such that

$$f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$$

is smooth on $\phi(U) \in \tau_{\mathbb{R}^n}$.

Theorem 2.1 Show that definition 1.4 is independent of the choice of chart. That is, if f is smooth on M some smooth $n-$manifold, then $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$ is smooth for *any* smooth chart (ϕ, U) on M.

Proof. Let M be a smooth $n-$manifold and suppose

$$f : M \to \mathbb{R}^k$$

is smooth. Let (ϕ, U) be an arbitrary smooth chart on M. We aim to show

$$f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$$

is smooth. Since f is smooth, there exists a chart, say (ψ, V) on M with $p \in V$ so that

$$f \circ \psi^{-1} : \psi(V) \to \mathbb{R}^k$$

is smooth. Since transition maps between charts are smooth, we know

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is smooth. But then

$$(f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1}) = f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$$

is smooth and $p \in V$ was arbitrary and thus choice of charts is independent. $\square$

Now for smooth maps between manifolds.

Definition 2.2 Let M, N be smooth manifolds and

$$F : M \to N$$

be any mapping. We say F is a *smooth map between manifolds* if for every $p \in M$, there exists smooth charts $(\phi, U), (\psi, V)$ on M, N respectively with $p \in U, F(p) \in V$ such that $F(U) \subseteq V$ and

$$\phi \circ F \circ \phi^{-1} : \phi(U) \to \phi(V)$$

is smooth.

§ 2.2: Some Properties of Smooth Maps

In this section we discuss properties & examples of smooth maps. From closures to relationship with continuity.

Proposition 2.1: Properties of smooth maps Let M, N, P be smooth manifolds with $F : M \to N, G : M \to N$ smooth maps. Then

1. $F \pm G$ is smooth.
2. kF is smooth for any scalar k.
3. $F \circ G$ is smooth if G is smooth at $p \in M$ and F is smooth at $G(p) \in N$.
4. FG is smooth.
5. $\frac{F}{G}$ is smooth, provided $G \neq 0$ on its domain.

Proof. We will only prove (3). Let M, N, P be smooth manifolds with or without boundary and let

$$G : M \to N, F : N \to P$$

be smooth. I claim $F \circ G$ is smooth. Let $p \in M$. By smoothness of F there exists smooth charts $(\phi, U), (\psi, V)$ on N, P respectively containing $G(p), F(G(p))$ respectively with $F(U) \subset V$ such that

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is smooth. Since G is continuous, there is a smooth chart (θ, W) on M containing p, then

$$\begin{aligned}\psi \circ (F \circ G) \circ \theta^{-1} &= (\psi \circ F \circ \phi^{-1}) \circ (\phi \circ G \circ \theta^{-1}) \\ &: \theta(W) \to \psi(V).\end{aligned}$$

Which is smooth as it is the composition of smooth spaces between Euclidean spaces. □

Theorem 2.2: Smooth implies continuous If F is a smooth map between manifolds, then F is continuous.

Proof. Let M, N be smooth manifolds and $F : M \to N$ any smooth map. Let $p \in M$, by smoothness of F, there exists a smooth chart (ϕ, U) on M with $p \in U$ and a smooth chart (ψ, V) on N with $F(p) \in V$ such that $F(U) \subseteq V$ and

$$\phi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is smooth and thus continuous. Since ϕ, ψ are homeomorphisms onto their images $\phi(U), \phi(V)$ respectively, this implies in turn that

$$F\big|_U := \psi^{-1} \circ (\psi \circ F \circ \phi^{-1}) \circ \phi : U \to V$$

is continuous as it is a composition of continuous, $F\big|_U$ is thus continuous. And since it is continuous at a neighborhood of an arbitrary $p \in M$, F itself is continuous on all of M. □

Theorem 2.3: Smoothness is local Let M, N be smooth manifold (with or without boundary), and $F : M \to N$ any map, then

1. If for every $p \in M$ there is a smooth chart (ϕ, U) on M such that the restriction map $F\big|_U$ is smooth, then F is smooth.
2. Conversely, if F is smooth, then its restriction to any open subset of M is also smooth.

Example 2.1(a) Smooth maps Here are some examples of known smooth maps:

1. Let $\exp : \mathbb{R} \to \mathbb{S}^1$ be given via
$$t \mapsto e^{2\pi i t}.$$
This is a smooth map.
2. The mapping $\epsilon : \mathbb{R}^n \to \mathbb{T}^n$ (the range space is the n torus defines as $T^n = \mathbb{S}^1 \times \mathbb{S}^1$) given via
$$(x_1, x_2, ..., x_n) \mapsto (e^{2\pi i x_1}, e^{2\pi i x_2}, ..., e^{2\pi i x_n}),$$
is a smooth map.
3. If $\iota : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is the inclusion map and $\pi : \mathbb{R}^{n+1} \setminus \{\bar{0}\} \to \mathbb{RP}^n$ is the quotient map, then if you define
$$q : \mathbb{S}^n \to \mathbb{RP}^n$$
as the restriction of π to $\mathbb{S}^n \subset \mathbb{R}^{n+1} \setminus \{\bar{0}\}$, then q is smooth because $q = \pi \circ \iota$ which is a composition of smooth and is thus smooth by proposition 1.1.
4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given via $(x, y) \to x^2 + y^2$. Then f is smooth.

A few non examples now:

Example 2.1(b) Non-smooth maps We can have

1. If $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$ is defined via $x \mapsto |x|$, then at $x = 0$ we loose smoothness. Though this map is continuous, it looses differentiability at $x = 0$.
2. If $f : \mathbb{R} \to \mathbb{R}$ is defined via
$$x \mapsto \begin{cases} x & ; x \geq 0 \\ 2x & : x < 0 \end{cases}.$$
3. Consider $f : \mathbb{R} \to \mathbb{R}$ defined via
$$x \mapsto \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & ; x \neq 0 \\ 0 & : x = 0 \end{cases}.$$
This is continuous everywhere but the derivative is undefined at $x = 0$.

So smooth maps in differential geometry are analogous to continuous maps in topology. In the next section, we make the stronger assumptions for the analogy of a homeomorphism in point-set topology. Some properties are:

Proposition 2.2: Properties of Smooth maps, More Let M, N be smooth manifolds with or without boundary. Then

1. Every constant map is smooth.
2. The identity map on M is smooth.
3. If $\iota : U \hookrightarrow M$ is the inclusion map, then ι is smooth.

§ 2.3: Diffeomorphisms

Much like in topology we have two topological spaces having a continuous mapping, then the stronger notion of them being homoemorphic (where homeomorphic means 1-1, onto and continuous function with continuous inverse), in differential geometry, one has first smooth then the stronger notion of two manifolds being diffeomorphic.

Definition 2.3: Diffeomorphism Let M, N be smooth manifolds. We say M, N are *diffeomorphic*, often written $M \cong N$ if there exists a 1-1, onto smooth map $F : M \to N$ with smooth inverse. (It should be noted that there exists smooth maps between manifolds of different dimension but no diffeomorphism between maps with different dimension, see Example 1.4.4.).

Example 1.5(a) We have

1. Consider
$$F : \mathbb{B}^n \to \mathbb{R}^n, G : \mathbb{R}^n \to \mathbb{B}^n$$
given via
$$x \mapsto \frac{x}{\sqrt{1-|x|^2}}, y \mapsto \frac{y}{\sqrt{1+|y|^2}}.$$
It is clear to see $F = G^{-1}, G = F^{-1}$ and they are both smooth and 1-1, onto thus $\mathbb{B}^n \cong \mathbb{R}^n$.
2. For any $n-$manifold M and $U \subseteq M$ open, the associated homeomorphism (as (ϕ, U) is a chart on M)
$$\phi : U \to \tilde{U}$$
is a homeomorphism so for any $U \in \tau_M$, $U \cong \tilde{U}$ where $\tilde{U} \in \tau_{\mathbb{R}^n}$.
3. Every linear (affine) function, i.e., of the form
$$f(x) = mx + b$$
is a diffeomorphism from $\mathbb{R}$ onto itself.

Next, for some non-examples:

Example 1.5(b) (Non-Examples of Diffeomorphisms) A few non-examples of diffeomorphisms include, but are not limited to:

1. Consider $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$ via $x \mapsto x^2$ is certainly smooth but not a diffeomorphism since $f^{-1} = \sqrt{x}$ which has an undefined derivative at $x = 0$.
2. The inclusion map $\iota : (0, 1) \hookrightarrow \mathbb{R}$ is smooth but not diffeomorphic. Though it is 1-1, it is clearly not onto since it is clear to see that for instance $1 \in \mathbb{R}$ doesn't get "hit".
3. The projection mapping $\pi : \mathbb{R}^2 \to \mathbb{R}$ via $(x, y) \mapsto x$ is smooth and onto but not 1-1 thus no inverse exists even.
4. Any constant mapping is smooth but clearly not 1-1 so no inverse exists here either.

5. Any *covering map* is not a smooth mapping. You usually loose being 1-1 since covering maps need be surjective or onto.

Proposition 2.2: Properties of diffeomorphisms We have

1. Compositions of diffeomorphisms are diffeomorphisms.
2. Finite products of diffeomorphisms between smooth manifolds is a diffeomorphism.
3. Every diffeomorphism is a homeomorphism and an open map.
4. Being "Diffeomorphic" is an equivalence relation on the class of smooth manifolds.
5. The restriction of a diffeomorphism to a sub-manifold is again a diffeomorphism onto its image.

One may next beg the question, is the sum and difference of a diffeomorphism still a diffeomorphism? The short answer is no, to see this consider our next example:

Example 1.6 Note that diffeomorphisms need not be closed under sums, differences, or scalar multiplication. Consider $M = \mathbb{R}$ and put $f, g : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = x + 1, g(x) = x - 1,$$

then

$$f - g = 2$$

which is constant thus not invertible. In general there is no way to add or subtract on a manifold without some induced metric.

Theorem 2.4: Diffeomorphism Invariance of Dimension A non-empty $n-$manifold cannot be diffeomorphic to an $m-$manifold unless $m = n$.

Proof. Suppose M is a non-empty smooth $m-$manifold and N a non-empty smooth $n-$manifold. Let $F : M \to N$ be a smooth map. Choose $p \in M$ and let $(\phi, U), (\psi, V)$ be smooth charts containing $p, F(p)$ respectively. Then $\psi \circ F \circ \phi^{-1}$ is a diffeomorphism from some open subset of $\mathbb{R}^m$ to and open subset of $\mathbb{R}^n$ and since we mentioned previously these are homeomorphic if and only if $m = n$ (proving this we won't get to until later in these notes, the part on *de Rham cohomology*) thus $m = n$. □

Theorem 2.5: Diffeomorphism Invariance of boundary Suppose M, N are smooth manifolds with boundary, then

$$F(\partial M) = \partial N.$$

And the restriction of F to the interior of M maps to the interior of N.

For the proof, note that if $p \in M$ with boundary, there is a smooth chart(ϕ, U) on M such that $\phi(p) \in \mathbb{H}^n$ the closed upper half plane thus $F(\phi(p)) \in \overline{\mathbb{H}^n}$. And if $\phi(p)$ is an interior point, then $F(\phi(p)) \in \text{Int } \mathbb{H}^n$.

A brief note on the unique structures on $\mathbb{R}^n$*:*

If $n \neq 4$, then there exists a unique smooth structure on $\mathbb{R}^n$ up to diffeomorphism. In the case $n = 4$, there are uncountably infinite smooth structures on can place on $\mathbb{R}^4$ none of which are diffeomorphic to each other. These are called "exotic $\mathbb{R}^4$'s and were proved by Donaldson & Freedman in 1984. The reason being *Whitney Trick*, which fails in dimension $n = 4$ but this is out of the scope of these notes, fails in dimension 4. So if $n \neq 4$, then there exists a unique, smooth structure on $\mathbb{R}^n$.

Just a bit of extra known math facts: Analogously, there exists $n-$spheres which are homeomorphic to $\mathbb{S}^n$ for some n but not diffeomorphic to it. These are called *Exotic spheres*. The lowest dimensions and first exotic sphere which was discovered has dimensions $n = 7$, so there exists a $7-$sphere which is homeomorphic (topological sense) to $\mathbb{S}^7$ but is not diffeomorphic (smooth sense) to it. It was first introduced by John Milnor in 1956 as $\mathbb{S}^3$ *bundles* over $\mathbb{S}^4$.

§ 2.4: Partitions of unity

Exercise 2.1 The function $f : \mathbb{R} \to \mathbb{R}$ via

$$t \mapsto \begin{cases} e^{-\frac{1}{t}} & ; t > 0 \\ 0 & ; t \leq 0 \end{cases},$$

is smooth.

Lemma 2.1 Given any $r_1 < r_2 \in \mathbb{R}$, there exists a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $h(t) \equiv 1$, for $t \leq r_1$, $h(t) \in (0,1)$, for $t \in (r_1, r_2)$ and $h(t) \equiv 0$ for $t \geq r_2$. That is,

$$h(t) := \begin{cases} 1 & ; t \leq r_1 \\ 0 & ; t \geq r_2 \\ h(t) \in (0,1) & ; t \in (r_1, r_2) \end{cases}.$$

To see this let f be as the previous exercise and put

$$h(t) := \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}.$$

This h is said to be a *cutoff function*.

Lemma 2.2 Given $r_1 < r_2 \in \mathbb{R}$, there exists a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ such that $H \equiv 1$ on $\bar{B}_{r_1}(0)$ and $H(x) \in (0,1)$ for all $x \in B_{r_2}(0) \setminus \bar{B}_{r_1}(0)$ and $H \equiv 0$ on $\mathbb{R}^n \setminus B_{r_2}(0)$.

To see this take $H(x) := h(|x|)$. This H is what is known as a *bump function*.

Definition 2.4: Let $f : M \to \mathbb{R}$ where M is a topological space, the *support of f* is

$$\text{supp } f = \overline{\{p \in M : f(p) \neq 0\}}.$$

That is, the topological closure of the points who have non-zero value under f. A function is said to have *compact support* if its support is a compact set.

The next construction is crucial in many fields:

Definition 2.5 Let M be a topological space and $\mathcal{X} = \{X_a\}_{a \in A}$, A some arbitrary indexing set. A *partition of unity subordinate to $\mathcal{X}$* is an indexed family $\{\psi_a\}_{a \in A}$ of continuous functions $\psi_a : M \to \mathbb{R}$ such that

1. $\psi_a(x) \in [0,1]$ for every $a \in A$ and every $x \in M$.
2. $\sum_{a \in a} \psi_a(x) = 1$ for all $x \in M$.
3. The family of supports $\{\text{supp } \psi_a\}_{a \in A}$ is *locally finite collection*. Meaning every point has a neighborhood that intersects $\text{supp}\psi_a$ for only finite many values of a.
4. Lastly, one has that supp $\psi_a \subseteq X_a$ for each $a \in A$. That is, for each $a \in A$, the support of ψ_a is fully contained in X_a.

Theorem 2.6: Existence of partitions of unity Let M be a smooth manifold (with or without boundary), and if $\mathcal{X} = \{X_a\}_{a \in A}$ is an indexed open cover for M, then there exists a partition of unity subbordinate to $\mathcal{X}$.

The proof of this uses the H bump function we previously defined. Some applications of partitions of unity include proving existence of exhaustion functions.

Definition 2.6 If M is a topological space, an *exhaustion function for* M is a continuous function $f : M \to \mathbb{R}$ if the *sub-level set*, $f^{-1}((-\infty, x_0])$ is compact in M for each $x_0 \in \mathbb{R}$. I.e., for

$$f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}$$

via

$$f(x) := |x|^2, g(x) := \frac{1}{1 - |x|^2}$$

are smooth exhaustion functions. If M is compact, then every continuous real valued function is an exhaustion function. So this is only interesting for non-compact manifolds.

Theorem 2.7: Existence of Smooth Exhaustion functions To prove this requires partitions of unity which we leave to the reader.

Theorem 2.8: Level sets of smooth functions Let M be any smooth manifold. If $K \subseteq M$ is closed, then there exists a smooth nonnegative function $f : M \to \mathbb{R}$ such that

$$f^{-1}(0) = K.$$

Proof. We prove the special case where $M = \mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$ is closed. As K is closed, its complement is open thus for each $x \in \mathbb{R}^n \setminus K$, there exists some $\epsilon \in (0, 1]$ such that

$$B_\epsilon(x) \subseteq \mathbb{R}^n \setminus K.$$

As $\mathbb{R}^n \setminus K$ is open, we can write it as the union

$$B = \bigcup_j B_{\epsilon_j}(x_j).$$

Let

$$h : \mathbb{R}^n \to \mathbb{R}$$

be a smooth bump function which is equal to 1 on $\overline{B}_{\frac{1}{2}}(0)$ supported in $B_1(0)$. For each $j \in \mathbb{Z}^+$, let $C_j \geq 1$ be a constant which bounds absolute values of h and all of its partials up to order j. Define

$$f : \mathbb{R}^n \to \mathbb{R}$$

via

$$x \mapsto \sum_j \frac{(\epsilon_j)^j}{2^j C_j} h\left(\frac{x - x_j}{\epsilon_j}\right).$$

From calculus, this series converges uniformly to some continuous function be the Weierstrass $M-$test. Since the $j-$th term is positive exactly when

$$x \in B_{\epsilon_j}(x_j),$$

we have that $f \equiv 0$ on K and positive elsewhere. All that is left is to show f is smooth. We leave these details to the reader but the trick is to use the chain rule and induct on the order of derivative. ☐

§ 3. Tangent Vectors & Tangent Spaces

§ 3.1: Tangent Vectors

We first begin with explaining tangent vectors of Euclidean space. We then generalize this from Euclidean space into arbitrary smooth manifolds. We keep in mind a separate copy of $\mathbb{R}^n$ at each point.

Definition 3.1 Given $a \in \mathbb{R}^n$, the *geometric tangent space to* $\mathbb{R}^n$ *at* a, denoted $\mathbb{R}^n_a$, by

$$\mathbb{R}^n_a := \{a\} \times \mathbb{R}^n := \{(a,v) : v \in \mathbb{R}^n\}.$$

Elements of this space we denote v_a. We also have that

$$(v+w)_a = v_a + w_a, c(v_a) = (cv)_a.$$

And if $a \neq b$ then $\mathbb{R}^n_a \cap \mathbb{R}^n_b = \emptyset$. the issue here is that $\mathbb{R}^n_a \cong \mathbb{R}^n$. SO for $v_a \in \mathbb{R}^n_a$ we get a map

$$D_v\big|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R},$$

which takes the directional derivative:

$$D_v\big|_a f = D_v f(a) = \frac{d}{dt}\big|_{t=0} f(a+tv).$$

The operation is linear over $\mathbb{R}$ and satisfies the product rule, quotient rule, and chain rule. Namely, the chain rule is (given $v_a = v_i e_i\big|_a$ where e_i is the standard basis vector)

$$D_v\big|_a f = v_i \frac{\partial f}{\partial x_i}(a).$$

With this construction in mind we have *Definition 3.2*, if $a \in \mathbb{R}^n$ and $w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a *derivation at* a if it is $\mathbb{R}-$linear and satisfies the product rule:

$$w(fg) = wfg(a) + wgf(a).$$

Let $T_a\mathbb{R}^n$ denote the set of all derivations of $C^\infty(\mathbb{R}^n)$ at a. Clearly this is a vector space under the operations

$$(w_1 + w_2)f = w_1 f + w_2 f, \qquad c(wf) = (cw)f.$$

Moreover, $T_a\mathbb{R}^n \cong \mathbb{R}^n_a$ defined above. Some properties are in the following lemma:

Lemma 3.1: Properties of derivations Let $a \in \mathbb{R}^n$, $w \in T_a\mathbb{R}^n$, and $f, g \in C^\infty(\mathbb{R}^n)$. Then

1. If f is constant, then $wf = 0$.
2. If $f(a) = g(a) = 0$, then $w(fg) = 0$.

To prove this lemma we merely invoke the product rule. This lemma is used to prove the following proposition:

Proposition 3.1 Let $a \in \mathbb{R}^n$. Then

1. For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map $D_v\big|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined above is a derivation at a.
2. The map $v_a \mapsto D_v\big|_a$ shows $\mathbb{R}^n_a \cong T_a\mathbb{R}^n$.

Since we claim $T_a\mathbb{R}^n$ is a vector space, we must give it an explicit basis. This comes in the form of the following corollary:

Corollary 3.1: For any $a \in \mathbb{R}^n$, the n derivations

$$\left.\frac{\partial}{\partial x_1}\right|_a, \left.\frac{\partial}{\partial x_2}\right|_a, \ldots, \left.\frac{\partial}{\partial x_n}\right|_a$$

defined by

$$\left.\frac{\partial}{\partial x_j}\right|_a f := \frac{\partial f(a)}{\partial x_j}$$

forms a basis for $T_a\mathbb{R}^n$ and thus has dimension n.

All of this talk about tangent space of Euclidean space can be easily transferred over to defining the tangent space of some smooth manifold in the following sense:

Definition 3.2 Let M be a smooth manifold, with or without boundary with $p \in M$. A linear map $v : C^\infty(M) \to \mathbb{R}$ is called a *derivation at* p if it satisfies the product rule:

$$v(fg) = vfg(p) + f(p)vg \qquad \text{for all } f, g \in C^\infty(M).$$

The set of all derivations at p, denoted T_pM, is the *tangent space to* M *at* p. Elements are referred to as *tangent vectors at* p. We ends this sections with a brief lemma analogous to the one for derivations on a Euclidean space.

Lemma 3.2: Properties of Derivations on Manifolds Suppose M is a smooth manifold with or without boundary with $p \in M$, $v \in T_pM$ and $f, g \in C^\infty(M)$. Then

1. If f is a constant functions, then $vf = 0$.
2. If $f(p) = g(p) = 0$, then $v(fg) = 0$.

§ 3.2: Differential of a Smooth map

In this section we discuss how to take the notion of differentials in Euclidean space and transfer this on over to the language of manifolds. In Euclidean spaces, the total derivative of a map at a point (represented by its Jacobian matrix) is a linear map with the best "linear approximation" to the map near the given point. So for manifolds, we cannot discuss linear mapping between two smooth arbitrary manifolds but rather the map between their respective tangent spaces. This gives rise to our first definition in this section:

Definition 3.3 Let M, N be smooth manifolds with or without boundary. Let $F : M \to N$ be a smooth map. For each $p \in M$ define

$$dF_p : T_pM \to T_{F(p)}N$$

called the *differential of* F *as* p as follows. Given $v \in T_pM$ we let $dF_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by

$$dF_p(v)f := v(f \circ F).$$

Note if $f \in C^\infty(N)$ then $f \circ F \in C^\infty(M)$ so $v(f \circ F)$ makes sense. Moreover, the operator $dF_p(v) : C^\infty(N) \to \mathbb{R}$ because v is and is a derivation at $F(p)$ because for any $f, g \in C^\infty(N)$, one has

$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) \\ &= v((f \circ F)(g \circ F)) \\ &= v(f \circ F)g \circ F(p) + f \circ F(p)v(g \circ F) \\ &= g(F(p))dF_p(v)f + f(F(p))dF_p(v)g. \end{aligned}$$

Proposition 3.2: Properties of differentials Let M, N, P be smooth manifolds with or without boundary and $F : M \to N, G : N \to P$ be smooth maps with $p \in M$, then

1. Linearity: $dF_p : T_pM \to T_{F(p)}N$ is linear.
2. Compositions: $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$.
3. Identity mapping: $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.
4. Preservation under diffeomorphisms: If $F : M \to N$ is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism and

$$(dF_p)^{-1} = d(F^{-1})_{F(p)}.$$

Furthermore, the dimension of the tangent space is the same dimensional vector space as its corresponding dimensional manifold. Also, tangent vectors act locally, in that:

Proposition 3.3: Locally agreeing functions Let M be a smooth manifold with or without boundary, $p \in M, v \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood of p, then $vf = vg$.

Proof. As $f, g \in C^\infty(M)$, then

$$h := f - g \in C^\infty(M).$$

Furthermore, h is a smooth bump functions who vanishes on some neighborhood of p. That is if

$$p \in U,$$

for some $U \in \tau_M$,then

$$h\big|_U \equiv 0.$$

Let $\psi \in C^\infty(M)$ be a smooth bump function with $\psi \equiv 1$ on supp h and is supported in $M \setminus \{p\}$. Thus the product $\psi h \equiv h$. And since $\psi(p) = h(p) = 0$, then by Lemma 3.2,

$$v(h) = v(\psi h) = 0.$$

And by linearity, $vf = vg$. □

Given any open subset $U \subseteq M$, the isomorphism $d\iota_p$, where ι_p is the inclusion at p, from $T_pU \to T_pM$ is canonically defined independent of any choices so we identify $T + pU$ with T_pM. So tangent vectors v can be unambiguously identified with functions defined on some neighborhood of p and note all of M, that is tangent vectors are represented locally not globally.

Proposition 3.4: Dimension of Tangent space For each $p \in M$ a smooth $n-$manifold, T_pM is an $n-$dimensional vector space.

Proof. Given $p \in M$ let (ϕ, U) be a smooth chart on M containing p. By Proposition 3.2.4, since ϕ is a homeomorphism from

$$U \to \tilde{U} \in \tau_{\mathbb{R}^n},$$

we have

$$d\phi_p : T_pU \to T_{\phi(p)}\tilde{U}$$

is an isomorphism. Since $T_pU \cong T_pM$ and $T_{\phi(p)}\tilde{U} \cong T_{\phi(p)}\mathbb{R}^n$, it follows that

$$\dim\, T_pM = \dim\, T_{\phi(p)}\mathbb{R}^n = n.$$

□

We end this section with a proposition of a (finite) product of smooth $n-$manifolds. The identification is the tangent space of a finite product of distinct manifolds say

$$M = \prod_{j=1}^{N} M_j.$$

Proposition 3.5: Tangent spae of a finite product Let $M = \prod_{j=1}^N$ where M_j is smooth, with or without boundary, for each $j \in \{1, 2, ..., N\}$. For each j, let

$$\pi_j : M \to M_j$$

be the projection onto the M_j-th factor. For any $p = (p_1, p_2, ..., p_N) \in M$, the map

$$\alpha : T_p(M) \to \bigoplus_{j=1}^{N} T_p(M_j),$$

defined via

$$v \mapsto (d(\pi_1)_p(v), d(\pi_2)_p(v), ..., d(\pi_N)_p(v))$$

is an isomorphism.

§ 3.3: Computations in Coordinates

We next move onto how to do computations with tangent vectors and differentials in local coordinates. First suppose M is a smooth manifold and (ϕ, U) is a smooth chart on M (so $U \in \tau_M$). Well then

$$\phi : U \to \tilde{U} \in \tau_{\mathbb{R}^n}$$

is a homeomorphism. Combining linearity of the differential together with sub-manifold tangent space definitions, one has that

$$d\phi_p : T_pM \to T_{\phi(p)}\mathbb{R}^n$$

is an isomorphism of tangent spaces in the vector space sense. By Corollary 3.1, the derivations

$$\left.\frac{\partial}{\partial x_1}\right|_{\phi(p)}, \left.\frac{\partial}{\partial x_2}\right|_{\phi(p)}, ..., \left.\frac{\partial}{\partial x_n}\right|_{\phi(p)}$$

form a basis for $T_{\phi(p)}\mathbb{R}^n$. Thus the pre-image of these vectors under $d\phi_p$ form a basis for T_pM. We use the notation $\left.\frac{\partial}{\partial x_j}\right|_p$ for these vectors with either of the following properties:

$$\begin{aligned}\left.\frac{\partial}{\partial x_j}\right|_p &= (d\phi_p)^{-1}\left(\left.\frac{\partial}{\partial x_j}\right|_{\phi(p)}\right) \\ &= d(\phi^{-1})_{\phi(p)}\left(\left.\frac{\partial}{\partial x_j}\right|_{\phi(p)}\right).\end{aligned}$$

Unraveling these definitions, we see that $\left.\frac{\partial}{\partial x_j}\right|_p$ acts on $f \in C^\infty(U)$ via

$$\begin{aligned}\left.\frac{\partial}{\partial x_j}\right|_p f &= \left.\frac{\partial}{\partial x_j}\right|_{\phi(p)}(f \circ \phi^{-1}) \\ &= \frac{\partial \tilde{f}}{\partial x_j}(\tilde{p}).\end{aligned}$$

Here, $\tilde{f} = f \circ \phi^{-1}$ is the coordinate representation of f and $\tilde{p} = (p_1, p_2, ..., p_n) = \phi(p)$ is the coordinate representation of p. We will see shortly that when integrating differential forms on manifolds, the choice of integration is independent of your choice of coordinate representation which is the beauty in this subject field.

Definition 3.4 The vectors $\left.\frac{\partial}{\partial x_j}\right|_p$ are called *coordinate vectors at p* associated with the given coordinate system. In the special case of the standard coordinates on $\mathbb{R}^n$ they are merely the $j-$the partial derivatives we have seen in calculus 3. If M is a manifold with boundary and $p \in \text{Int } M$, i.e., is an interior point, things run smooth, but if $p \in \partial M$, then one must substitute the entire Euclidean space with the upper half plane. The details we have been discussing thus far can be summarized into the following proposition:

Proposition 3.6 Let M be a smooth manifold with or without boundary with $p \in M$. Then T_pM is an $n-$dimensional vector space and for any smooth chart containing p, then the vectors

$$\left.\frac{\partial}{\partial x_1}\right|_p, \left.\frac{\partial}{\partial x_2}\right|_p, ..., \left.\frac{\partial}{\partial x_n}\right|_p$$

form a basis form T_pM.

Thus a tangent vector $v \in T_pM$ can be uniquely written as a linear combination

$$v = v_j \frac{\partial}{\partial x_j}\bigg|_p.$$

Definition 3.5 The ordered basis $\left(\frac{\partial}{\partial x_j}\big|_p\right)$ is the *coordinate basis for* T_pM, and the numbers $(v_1, v_2, ..., v_n)$ are the *components of* v with respect to the coordinate basis. If v is known, its components can be computed directly. For each j, the components of v are given by $v_k = v(x_k)$ (here we think of x_k as a smooth real valued function on U) because

$$\begin{aligned} v(x_k) &= \left(v_j \frac{\partial}{\partial x_j}\bigg|_p\right)(x_k) \\ &= v_j \frac{\partial x_k}{\partial x_j}(p) \\ &= v_k. \end{aligned}$$

Next, we discuss the what differentials in look like in coordinates. First consider $F : U \to V$ where $U \in \tau_{\mathbb{R}^n}, V \in \tau_{\mathbb{R}^m}$. For a given $p \in U$ we determine the matrix of

$$dF_p : T_pU \to T_{F(p)}V$$

in terms of the standard coordinate basis. We let $(x_1, x_2, ..., x_n) \in \mathbb{R}^n, (y_1, y_2, ..., y_m) \in \mathbb{R}^m$ and use the chain rule as follows:

$$\begin{aligned} dF_p\left(\frac{\partial}{\partial x_j}\bigg|_p\right)f &= \frac{\partial}{\partial x_j}\bigg|_p (f \circ F) \\ &= \frac{\partial f}{\partial y_k}\Big(F(p)\Big)\frac{\partial F_k}{\partial x_j}(p) \\ &= \left(\frac{\partial F_k}{\partial x_j}(p)\frac{\partial}{\partial y_k}\bigg|_{F(p)}\right)(p). \end{aligned}$$

Thus

$$dF_p\left(\frac{\partial}{\partial x_j}\bigg|_p\right) = \frac{\partial F_k}{\partial x_j}(p)\frac{\partial}{\partial y_k}\bigg|_{F(p)} \qquad (*).$$

In other words, the matrix of dF_p in terms of the coordinate basis is

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1}(p) & ... & \frac{\partial F_1}{\partial x_n}(p) \\ .. & .. & .. \\ .. & .. & .. \\ \frac{\partial F_m}{\partial x_1}(p) & ... & \frac{\partial F_m}{\partial x_n}(p) \end{pmatrix}.$$

Which is precisely the Jacobian of the matrix representing F at p. Which is the matrix of the total derivative $DF(p) : \mathbb{R}^n \to \mathbb{R}^m$. Therefore, in our case, dF_p corresponds to the total derivative $DF(p)$ under the usual identification of a Euclidean space with its tangent space.

Now consider some smooth map $F : M \to N$ between manifolds with or without boundary. If we choose smooth coordinate chart (ϕ, U) on M containing p and (ψ, V) on N containing $F(p)$, we obtain the coordinate representation

$$\tilde{F} = \psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(V).$$

Let $\tilde{p} = \phi(p)$ denote the coordinate representation of p. By the computation above, $d\tilde{F}_{\tilde{p}}$ is represented with respect to the standard coordinate basis by the Jacobian matrix of $\tilde{F}$ at $\tilde{p}$. And since $F \circ \phi^{-1} = \tilde{F} \circ \psi^{-1}$, we have

$$\begin{aligned} dF_p\left(\left.\frac{\partial}{\partial x_j}\right|_p\right) &= dF_p\left(d(\phi^{-1})_{\tilde{p}}\left(\left.\frac{\partial}{\partial x_j}\right|_{\tilde{p}}\right)\right) \\ &= d(\psi^{-1})_{\tilde{F}(\tilde{p})}\left(d\tilde{F}_{\tilde{p}}\left(\left.\frac{\partial}{\partial x_j}\right|_{\tilde{p}}\right)\right) \\ &= d(\psi^{-1})_{\tilde{F}(\tilde{p})}\left(\frac{\partial \tilde{F}_k}{\partial x_j}(\tilde{p})\left.\frac{\partial}{\partial y_k}\right|_{\tilde{F}(\tilde{p})}\right) \\ &= \frac{\partial \tilde{F}_k}{\partial x_j}(\tilde{p})\left.\frac{\partial}{\partial y_k}\right|_{F(p)}. \end{aligned}$$

Definition 3.6 Thus dF_p is represented in coordinate basis by the Jacobian of the coordinate representative of F. In keeping in spirit with the language of differential geometry, the *differential*, often called the *tangent map*, the *total derivative* or *derivative of* F. Since it "pushes" vectors (point-wise) forward from domain to co-domain, it is also referred to as the *pushforward*, and often denoted:

$$F'(p), Df, dF_p, F_*(p), DF(p).$$

We will use dF_p as the differential and $DF(p)$ as the total derivative. And to be more clear, $DF(p) : \mathbb{R}^n \to \mathbb{R}^m$ whereas $dF_p : T_pM \to T_{F(p)}N$.

We now discuss the subject of a change of coordinates. Let $(\phi, U), (\psi, V)$ be two smooth charts on M with $p \in U \cap V$. Let use denote the coordinate functions of ϕ by (x_j) and those of ψ by $(\bar{x}_j)$. Any tangent vector to p can be represented by

$$\left(\left.\frac{\partial}{\partial x_j}\right|_p\right)$$

or by

$$\left(\left.\frac{\partial}{\partial \bar{x}_j}\right|_p\right).$$

So one might ask, how are these two related?

Here, it is usual to write the transition maps

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V),$$

in the following shorthand

$$\psi \circ \phi^{-1}\big(x\big) = \big(\bar{x}_1(x), \bar{x}_2(x), ..., \bar{x}_n(x)\big).$$

This may be abuse of notation as $\bar{x}_j : U \to \tilde{U}$ where $U \in \tau_M, \tilde{U} \in \tau_{\mathbb{R}^n}$. And x is to be thought of as a point. More specifically, $p \in \phi(U \cap V)$. Thus by $(*)$, the differential, $d(\psi \circ \phi^{-1})_{\phi(p)}$ can be written as

$$d(\psi \circ \phi^{-1})_{\phi(p)}\left(\frac{\partial}{\partial x_j}\bigg|_{\phi(p)}\right) = \frac{\partial \bar{x}_k}{\partial x_j}\big(\phi(p)\big)\frac{\partial}{\partial \bar{x}_k}\bigg|_{\psi(p)}.$$

Using the definition of coordinate vectors, we have

$$\begin{aligned}\frac{\partial}{\partial x_j}\bigg|_p &= d(\phi^{-1})_{\phi(p)}\left(\frac{\partial}{\partial x_j}\bigg|_{\phi(p)}\right)\\ &= d(\psi^{-1})_{\psi(p)} \circ d(\psi \circ \phi^{-1})_{\phi(p)}\left(\frac{\partial}{\partial x_j}\bigg|_{\phi(p)}\right)\\ &= d(\psi^{-1})_{\psi(p)}\left(\frac{\partial \bar{x}_k}{\partial x_j}\big(\phi(p)\big)\frac{\partial}{\partial \bar{x}_k}\bigg|_{\psi(p)}\right)\\ &= \frac{\partial \bar{x}_k}{\partial x_j}\big(\tilde{p}\big)\frac{\partial}{\partial \bar{x}_k}\bigg|_p \qquad (**).\end{aligned}$$

Where again, here $\phi(p) = \tilde{p}$. Applying this to the coordinates of a vector $v \in T_pM$, we find the components transform by

$$\bar{v}_k = \frac{\partial \bar{x}_k}{\partial x_j}\big(\tilde{p}\big)v_j.$$

Example 3.1: Polar coordinates The transition map between polar and standard form is given via

$$(x, y) \mapsto (r\cos\theta, r\sin\theta).$$

Let $p \in \mathbb{R}^2$ have polar representation

$$(r, \theta) = \big(2, \frac{\pi}{2}\big).$$

Let $v \in T_p\mathbb{R}^2$ be the tangent vector whose coordinate representation is given by

$$v = 3\frac{\partial}{\partial r}\bigg|_p - \frac{\partial}{\partial \theta}\bigg|_p.$$

Applying $(**)$ to the coordinate vectors,

$$\begin{aligned}\frac{\partial}{\partial r}\bigg|_p &= \cos\big(\frac{\pi}{2}\big)\frac{\partial}{\partial x}\bigg|_p + \sin\big(\frac{\pi}{2}\big)\frac{\partial}{\partial y}\bigg|_p\\ &= \frac{\partial}{\partial y}\bigg|_p.\end{aligned}$$

And

$$\begin{aligned}\frac{\partial}{\partial\theta}\bigg|_p &= -2\sin\left(\frac{\pi}{2}\right)\frac{\partial}{\partial x}\bigg|_p + 2\cos\left(\frac{\pi}{2}\right)\frac{\partial}{\partial y}\bigg|_p \\ &= -2\frac{\partial}{\partial x}\bigg|_p .\end{aligned}$$

Thus v is given by

$$v = 3\frac{\partial}{\partial x}\bigg|_p + 2\frac{\partial}{\partial y}\bigg|_p .$$

Exercise 3.1 Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Let

$$z_x = x, z_y = y + x^3.$$

Show (z_x, z_y) are smooth coordinate on $\mathbb{R}^2$ and if $p \in \mathbb{R}^2$ is $(0, 1)$ then

$$\frac{\partial}{\partial x}\bigg|_p \neq \frac{\partial}{\partial z_x}\bigg|_p .$$

Proof. First, to show (z_x, z_y) is smooth. First note the inverse maps are

$$x = z_x, y = z_y - z_x^3.$$

And furthermore, the determinant of the Jacobian is $1 \neq 0$ so the coordinates are smooth. ☐

§ 4. Differential Forms

§ 4.1: 0 & 1 Forms